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Aeroelastic Response of Nonlinear Wing Sections Using a Functional Series Technique

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The determination of the subcritical aeroelastic response and flutter instability of nonlinear two-dimensional wing sections in an incompressible flowfield via indicial functions and Volterra series approach is considered. The aeroelastic governing equations are based on the inclusion of stiffness and damping nonlinearities in plunging and pitching and of the linear unsteady aerodynamics and consideration of an arbitrary time-dependent external pressure pulse. Nonlinear unsteady aeroelastic kernels are obtained, and based on these, time histories of the subcritical aeroelastic response are determined. Conclusions and results displaying the implications of the considered effects are supplied.

Nomenclature

a	=	dimensionless elastic axis position measured from the midchord, positive aft
$C(k)$, $F(k)$, $G(k)$	=	Theodorsen's function and its real and imaginary parts, respectively
$C_{L\alpha}$	=	lift-curve slope
c	=	chord length of two-dimensional wing section, $2b$
c_{hi}, $c_{\alpha i}$, K_{hi}, $K_{\alpha i}$	=	damping and stiffness coefficients in plunging and pitching ($i = 1,2,3$-linear, quadratic, cubic), respectively
h, ξ	=	plunging displacement and its dimensionless counterpart (h/b), respectively
h_n, H_n	=	nth order Volterra kernel in time and its Laplace transformed counterpart, respectively
I_α, r_α	=	mass moment of inertia per unit wing span and the dimensionless radius of gyration $(I_\alpha/mb^2)^{1/2}$, respectively
L_a, M_a	=	total lift and moment per unit span
L_b, l_b	=	overpressure of the N-wave shock pulse and its dimensionless counterpart, $(L_b b/mU_\infty^2)$, respectively
l_a, m_a	=	dimensionless aerodynamic lift and moment, $(L_a b/mU_\infty^2)$ and $(M_a b^2/I_\alpha U_\infty^2)$, respectively
m, μ	=	airfoil mass per unit length and reduced mass ratio, $(m/\pi\rho b^2)$, respectively
P_m, $\wp_m$	=	peak reflected pressure amplitude and its dimensionless counterpart $(P_m b/mU_\infty^2)$, respectively
r	=	shock pulse length factor
S_α, χ_α	=	static unbalance about the elastic axis and its dimensionless counterpart S_α/mb, respectively
s_j, $\mathscr{L}$	=	Laplace transform variable and Laplace operator, respectively, $s_j = ik_j$; $i = \sqrt{-1}$
t, τ	=	time variables and dimensionless counterpart $(U_\infty t/b)$, respectively
t_p, τ_p	=	positive phase duration, measured from the time of the arrival of the pulse, and its dimensionless value, respectively
U_∞, V	=	freestream speed and its dimensionless counterpart $(U_\infty/b\omega_\alpha)$, respectively
$x(t)$	=	time-dependent external pulse (traveling gust and wake blast wave)
$y(t)$	=	response in the considered degree of freedom (pitch α and/or plunge h)
α	=	twist angle about the pitch axis
ζ_h, ζ_α	=	structural damping ratios in plunging $(c_h/2m\omega_h)$ and pitching $(c_\alpha/2I_\alpha\omega_\alpha)$, respectively
ρ	=	air density
$\phi(\tau)$	=	Wagner's indicial function
ω, k	=	circular and reduced frequencies $(\omega b/U_\infty)$, respectively
ω_h, ω_α	=	uncoupled frequencies in plunging and pitching $(K_h/m)^{1/2}$ and $(K_\alpha/I_\alpha)^{1/2}$, respectively
$\bar{\omega}$	=	plunging–pitching frequency ratio (ω_h/ω_α)

Superscripts

$(\hat{\cdot})$	=	quantities in Laplace transformed space
$(\dot{\cdot})$, $(\cdot)'$	=	derivatives with respect to time t and the dimensionless time τ, respectively

I. Introduction

IT is a well-known fact that within the linearized approach of the aeroelasticity discipline it is possible to determine the divergence and the flutter instability boundaries and also the linearized subcritical aeroelastic response to time-dependent external pulses. On the other hand, the nonlinear approach of the aeroelastic problem can provide important information such as 1) the influence of the considered nonlinearities on the subcritical aeroelastic response and 2) whether the aeroelastic instabilities are benign or catastrophic. In other words, such an approach gives the possibility of determining in what conditions the flutter speed can be exceeded without

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the occurrence of a catastrophic failure of the wing (in which case the flutter is benign), as well as the conditions in which undamped oscillations might appear at velocities below the flutter velocity (in which case the flutter is catastrophic). In addition, the considered nonlinearities play a great role on the subcritical aeroelastic response of wing sections. Because of the strong implications of various nonlinearities on the aeroelastic response of highly flexible lifting surfaces, their related aeroelastic phenomena should be analyzed within a more encompassing context than in the standard linearized framework. Aircraft wing structures often feature nonlinearities, which affect their aeroelastic behavior and performance characteristics, and flutter boundaries. For these reasons, in order to investigate the aeroelastic behavior of lifting surfaces in the vicinity of the flutter boundary the aeroelastic governing equations need to include corresponding nonlinear terms.

The advantage of applying a technique based on Volterra's series[1−5] and indicial functions[6−8] consists, among others, in the possibility to investigate the nonlinear aeroelastic systems within a rigorous theoretical basis. For exhaustive treatments of the Volterra series concept applied in the structural dynamics, the interested reader is referred to the recent book by Worden and Tomlinson.[9] As a limiting case, based on the first-order Volterra kernel, the study of the linear aeroelastic stability of the systems can be carried out. This methodology can encompass the case of an arbitrary number of degrees of freedom and at the same time is conceptually clearer, computationally simpler, and can provide more accurate and realistic results as compared to the conventional techniques used in nonlinear aeroelastic systems based on perturbation and multiple-scale methods. In addition, as shown in this paper, this method features a faster convergence as compared to the other methodologies.[9] Moreover, this method does not experience the limitations, usually characterizing the other techniques such as the Hilbert transform,[9−11] developed to identify nonlinear systems from the first-order frequency response function (FRF) or the phase plane methods that can describe the motion as just a function of two variables. In contrast to methods that are suitable mainly to single-degree-of-freedom systems, the Volterra series approach overcomes the shortcomings facing the other methods and provides a foundation for understanding the issue of the exchange of energy between the different mode frequencies.

Toward the end of determining the nonlinear unsteady aeroelastic kernels, the harmonic probing algorithm, referred to as the method of growing exponentials advanced by Bedrosian and Rice[12] and Boyd et al.[13] and the multidimensional Laplace transform[14] will be used. In addition to the aeroelastic response and determination of the flutter instability boundary, Volterra series considered in conjunction with this nonlinear aeroelastic model can be used to study the conditions rendering the flutter boundary a benign or a catastrophic one.[15,16]

Moreover, when the closed-loop dynamic response of actively controlled lifting surface is analyzed, also the feedback control forces and moments should be included.[17−20] The Volterra series approach can be applied toward this control purpose as well.

Volterra's series approach provides a firm basis for the treatment of the nonlinear subcritical aeroelastic response, in the sense that it supplies an explicit relationship between the input (any type of time-dependent external pulses, i.e., blast load, sonic boom, gust loads) and its response. With the so-called Volterra kernel identification scheme, the modeling of an aeroelastic system using this approach becomes feasible. However, this methodology requires determination for each specific flight conditions of the corresponding nonlinear kernel of the Volterra series. For this reason, the recent interest in the modeling of unsteady nonlinear aerodynamics has been focused on the identification of Volterra kernels in the time domain[21−24] and in the frequency domain.[25] A number of fundamental contributions related to Volterra series[1,2] have been applied, mainly in electrical engineering.[3−5] The original studies on functional series by Volterra[1] have been continued in the works by Volterra himself and of those by Rugh,[3] Schetzen,[4] and Boyd.[5] These concepts have been mainly used in the general nonlinear system theory. Originally, the method of Volterra series and Volterra kernel identification was developed to identify the nonlinear behavior in electrical circuits. In the aerospace field fundamental contributions were brought by Silva,[21−24] who has shown that the method is also applicable to aeroelastic systems (aerodynamic reactions and forced structural model). These contributions have opened a very promising avenue toward modeling and approaching nonlinear aeroelastic systems.

The present investigation concerns the aeroelastic response of two-dimensional nonlinear wing sections exposed to an incompressible flowfield and subjected to an external pressure pulse.[26−28] Based on Volterra functional series approach,[1−5] pertinent information about the effects of nonlinearities on either the aeroelastic response in the subcritical flight speed regime and their implication on the flutter boundary are supplied.

II. Basic Concepts

Because the principle of superposition is not applicable to nonlinear systems and in order to account for the various types and numbers of inputs, a combination of transfer functions (TFs) is used. These TFs are generated using the multidimensional Laplace transform of the Volterra kernels via a Mathematica® code developed by the authors.[29] Our approach, intended to address the subcritical response of the nonlinear aeroelastic governing equations, is based on its exact representation as an infinite sum of multidimensional convolution integrals, the first one (i.e., the linear kernel) being analogous to the linear indicial aeroelastic function. The full nonlinear aeroelastic response will be composed of additional higher-order contributions. In the frequency domain if the nonlinear function governing a system is "smooth," then for small inputs the system must be asymptotically linear.[3] One of the key issues is to determine, corresponding to the considered type of structural, damping and aerodynamic nonlinearities, the pertinent Volterra kernels. When active control is implemented the corresponding kernels should be determined as well.

III. Theory

In an attempt to make the paper reasonably self-contained, several elements associated with the indicial functions and Volterra's series as applied to aeroelastic system will be supplied.

A. Indicial Theory and Aerodynamic Loads

Using the aerodynamic indicial functions corresponding to the transient aerodynamic reaction to a step pulse, the aerodynamic forces and moments induced in any maneuver and flight regime can be determined. Aerodynamic forces and moments acting on a rapidly maneuvering aircraft are, in general, nonlinear functions of the motion variables, their time rate of change, and the history of the maneuvering.[30] However, in this study the linear aerodynamic theory is adopted. Once the response of the system to a step change in one of the disturbing variables (i.e., the indicial response) is known, the indicial method permits determination of the response to an arbitrary schedule of disturbances. There is a critical value of the flight speed, referred to as the flutter speed, above which the steady motion becomes unstable. The behavior of an aeroelastic system in the vicinity of the flutter instability can be investigated only in a nonlinear framework. In this context the Hopf bifurcation analysis can provide important information about the aeroelastic behavior in the vicinity of the flutter boundary. In the case of supercritical Hopf bifurcation, finite amplitude oscillations in the postflutter speed range can occur, whereas in the case of the subcritical Hopf bifurcation oscillations with increasing amplitudes, even if the system operates before reaching the flutter speed, can emerge.[31−33]

We need to mention that within the nonlinear indicial theory[34] the response of a nonlinear system to an arbitrary input can be constructed by integrating a nonlinear functional that involves the knowledge of the time-dependent input and the kernel response. Whereas within the linear indicial theory the linear kernel or linear impulse response can be convoluted with the input to predict the output of a linear system, the nonlinear indicial theory constitutes a generalization of this concept. It can also be shown that the traditional Volterra–Wiener theory of nonlinear systems constitutes a subset of the nonlinear indicial theory. The nonlinear unsteady aerodynamics valid throughout the subsonic incompressible/compressible, transonic, and supersonic flight speed regimes can be used and determined via the use of nonlinear indicial functions in conjunction with the Volterra series approach. Within the linearized unsteady aerodynamics of compressible flight speeds, the monograph by Leishman[35] provides an excellent presentation of the state of the art of the indicial function concept.

B. Volterra Functional Series Theory

As it was shown,[3,4] within Volterra's series approach the full response in the time domain $y(t)$ of the nonlinear systems with memory can be cast as

$$y(t) = \sum_{k=0}^{\infty} y_k(t) \tag{1}$$

where $y_k(t)$ is expressed as

$$y_k(t) = \underbrace{\iiiint_{-\infty}^{\infty}}_{k \text{ times}} h_k(t-\tau_1, t-\tau_2, \ldots t-\tau_k) \prod_{i=1}^{k} x(\tau_i)\,\mathrm{d}\tau_i \tag{2}$$

By a change of variables, it is possible to express Eq. (2) in contracted form as

$$y_k(t) = \underbrace{\iiiint_{0}^{\infty}}_{k \text{ times}} h_k(\tau_1, \tau_2, \ldots \tau_k) \prod_{i=1}^{k} x(t-\tau_i)\,\mathrm{d}\tau_i \tag{3}$$

It is assumed that $x(t) = 0$ for $\tau < 0$, implying that the system is causal.

With this restriction, all of the integrals in the subsequent discussions are different from zero over the time range $[0, \infty)$. Restricting the development of Eq. (3) to the first three terms, one obtains

$$\begin{aligned} y(t) = &\int h_1(\tau_1)x(t-\tau_1)\,\mathrm{d}\tau_1 \\ &+ \iint h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \\ &+ \iiint h_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3 \\ &+ \cdots \end{aligned} \tag{4}$$

The response of the system can be expressed in the frequency domain as well. Volterra series is essentially a polynomial approximation of the system, extension of Taylor series to systems with memory, whereas Volterra kernels $h_i(s_i)$ are a direct extension of the impulse response concept of the linear system theory to multiple dimensions.[1,3–5] Consequently, a multidimensional analogue of the impulse response can be used to characterize a nonlinear system.[21–24]

Having in view that the memory of aeroelastic systems is not infinite and, at the same time, the time-dependent external excitations, such as impulse, gust, blast, and sonic-boom pressure signatures are nonpersistent (in the sense that their effect decay as time unfolds), it is possible to characterize a nonlinear aeroelastic system via Volterra series. This fact is reflected in the interpretation of Volterra kernels as higher-order impulse response functions, that is, $h(\tau_1, \ldots, \tau_n) \to 0$ as $\tau_1, \ldots, \tau_n \to \infty$. We will use the definition of the nonlinear transfer function (TF) or higher-order impulse response functions, namely,

$$\begin{aligned} H_n(s_1, s_2, \ldots, s_n) = &\iint \ldots \int h_n(\tau_1, \tau_2, \ldots, \tau_n) \\ &\times e^{-s_1\tau_1} e^{-s_2\tau_2} \cdots e^{-s_n\tau_n}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 \cdots \mathrm{d}\tau_n \end{aligned} \tag{5}$$

as well as of their inverted counterparts:

$$\begin{aligned} h_n(\tau_1, \tau_2, \ldots \tau_n) = &\left(\frac{1}{2\pi i}\right)^n \int_{\sigma_n - i\infty}^{\sigma_n + i\infty} \cdots \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \\ &\times H_n(s_1, s_2, \ldots, s_n) e^{s_1\tau_1} e^{s_2\tau_2} \cdots e^{s_n\tau_n}\,\mathrm{d}s_1\,\mathrm{d}s_2 \cdots \mathrm{d}s_n \end{aligned} \tag{6}$$

Once Volterra's kernels are known, the response of the nonlinear aeroelastic system can fully be identified. As demonstrated in the Schetzen works,[3,5] without loss of generality the kernels will be taken as symmetric, in the sense of $H_n(s_1, s_2, \ldots) = H_n(s_2, s_1, \ldots)$, with a similar relationship being valid for h_n as well.

If we focus our attention on the linear system, the Laplace transform $\mathscr{L}$ of the first term of Eq. (4) yields the familiar Laplace domain expression $Y(s) = H(s)X(s)$, where $Y(s)$, $H(s)$, $X(s)$ are the Laplace transforms of $y(\tau)$, $h(\tau)$, $x(\tau)$, respectively. $H(s)$ is the TF of the system. It is a well-known fact that for the linear system either the first TF or the first kernel in time $h(\tau)$ contain all of the information about the aeroelastic system. Moreover, if the system is linear, the external load is uniquely related to the response by a convolution integral. With the use of functional series, that is, the Volterra series, this functional representation can be extended to nonlinear systems. The comparison between the prediction of the linear aeroelastic responses of the two-dimensional wing section in incompressible flowfield based on the Volterra series approach [using Theodorsen's function $C(k)$] and on the exact solution based on convolution integrals [using Wagner's function $\phi(\tau)$] is presented in Fig. 1 ($m = 1$ kg; $c_{h1} = 0.1$ N/ms^{-1}; $k_{h1} = 10^2$ N/m; $C_{L\alpha} = 2\pi$;

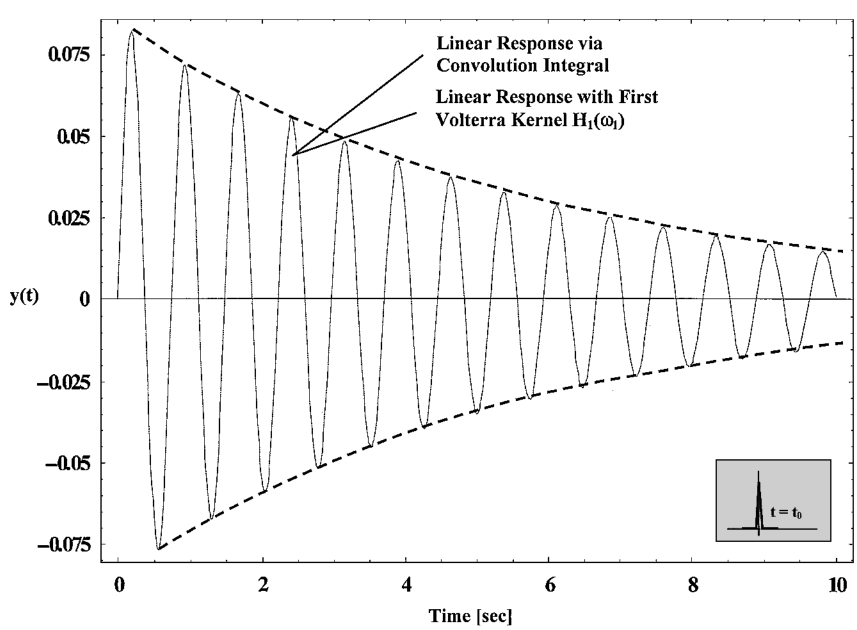

Fig. 1 Aeroelastic response to Dirac delta impulse, as represented in inset. Comparison of response prediction based on the first Volterra kernel and the exact solution.

$b = 1$m; $\rho = 1.225$ kg/m^3; $U_\infty = 1$ m/s). The indiscernible difference in the two response predictions assesses both the accuracy of the aeroelastic model and also the power of the methodology that combines Volterra's series and the indicial function.

IV. Mathematical Formulation

A. Structural and Damping Nonlinearities

The aeroelastic governing equation for one- and two-dimensional degree-of-freedom (DOF) wing sections that include the nonlinear stiffness and the damping in plunging and pitching will be analyzed next. In this sense, a one-DOF wing section featuring purely plunging motion and a two-DOF wing section featuring inertial and aerodynamic coupling in plunging h and pitching α will be analyzed. As already mentioned, the unsteady aerodynamic system is considered linear. A harmonic time-dependent external concentrated load is also included in the analysis. This can be considered to correspond, for example, to an engine mounted on an aircraft wing. As a characteristic of this approach, the TFs of the system would exist and would be the same for any excitations,[9,36–38] such as impulse, gusts, airblast, or sonic booms (random or deterministic ones). This is because TFs are independent of the input to the system, being a characteristic of the system itself. As a reminder, the validity of this method is based on the use of continuous polynomial-type nonlinearities. For nonlinear ordinary differential systems there are, in general, an infinite number of Volterra kernels. In practice, one can handle only a finite number of terms in the series, which leads to the problem of truncation accuracy. However, in the work by Wiener[2] it is suggested that the first terms of the series might be sufficient to represent the output of a nonlinear system if the nonlinearities are not too strong.

The use of the multidimensional Laplace transform as a function of several variables is a tool useful in stationary nonlinear system theory. The multivariable convolutions can be represented in terms of products of Laplace transforms. It is well known that the nonlinear aeroelastic systems cannot be described by a simple TF for two main reasons: 1) the response contains both the unsteady aerodynamic loads and the external excitation effects, and 2) in the nonlinear case the superposition principle is not applicable. It is also well known that any time-dependent external excitation, periodic or otherwise, can be represented to an arbitrary degree of accuracy by a sum of sinusoidal waves.[36] In this context, if the external load is expressed in terms of multiple sinusoidal forms (for example, a traveling gust load) this is easily expressed in exponential form, that is,

$$
\begin{aligned}
u(t) &= A\cos(\omega_A t) + B\cos(\omega_B t) \\
&= (A/2)(e^{s_A t} + e^{-s_A t}) + (B/2)(e^{s_B t} + e^{-s_B t}) \qquad (7)
\end{aligned}
$$

where $s_A = i\omega_A$ and $s_B = i\omega_B$. For clarity, it is convenient to adopt this approach for a single-DOF system. These results have more general bearing and can be extended for systems with multiple DOFs. In fact, by using the classical approach of the one-dimensional FRF it is possible to derive an analytical form of the multidimensional frequency response characteristics of nonlinear systems. The systems based on one DOF (plunging h) and two DOFs (pitching α and plunging h) will be considered in the next sections.

B. Plunging Airfoil Motion in an Incompressible Flowfield

The nonlinear equation of an airfoil featuring plunging motion can be expressed as

$$
m\ddot{h}(t) + \sum_{i=1}^{n}\left\{c_{\text{hi}}[\dot{h}(t)]^i + k_{\text{hi}}[h(t)]^i\right\} - L_a(t) = L_b(t) \qquad (8)
$$

where i defines the degree of the considered nonlinearity. In the numerical simulations i will assume the values 1,2,3, implying linear, quadratic, and cubic stiffness and damping nonlinearities. In addition, m is mass parameter and c_{hi}, k_{hi} are the damping and stiffness parameters associated with the damping and deflection in plunging corresponding for the ith power. In the right-hand-side member of these equations, $L_b(\tau)$ denotes the external time-dependent load acting on the rigid wing. In Eq. (8), the unsteady aerodynamic lift is represented as a function of the plunging degree of freedom as

$$
L_a(\tau) = -C_{L\alpha}\rho U_\infty^2 \int_{-\infty}^{\tau} \phi(\tau - \tau_0)h''\,\mathrm{d}\tau_0 - \frac{1}{2}\rho C_{L\alpha}U_\infty^2 h'' \qquad (9)
$$

The noncirculatory component present in Eq. (9) is represented in terms of convolution integral of the indicial Wagner's function.

To explain how this methodology works, let us determine, in terms of Volterra series, how a system responds to a harmonic or periodic time-dependent load. Let consider a periodic external excitation of the form

$$
L_b(t) = \sum_{j=1}^{n} X_j e^{s_j t} \qquad (10)
$$

The information acquired by the response to a harmonically time-dependent load can be used to obtain the response to any time-dependent external excitation. In fact, considering the case of a concentrated load arbitrarily located in the x, y plane of the wing we have

$$
u(x, y, t) = A\delta(x - x_0, y - y_0)e^{i\omega t} \qquad (11)
$$

where $\delta(\cdot)$, x_0, y_0, A, ω denote Dirac's distribution, location of the load, its amplitude, and excitation frequency, respectively. Once the TF is determined its counterpart in the time domain can be computed via the inverse Laplace transform $\mathcal{L}^{-1}$:

$$
\text{TF}(x, y, t) = \mathcal{L}^{-1}[\text{TF}(x, y, s)] = \frac{1}{2\pi i}\int_{\sigma_i - i\infty}^{\sigma_i + \infty} \text{TF}(x, y, s)e^{st}\,\mathrm{d}s \qquad (12)
$$

The general procedure to identify the aeroelastic kernels of various order $(1, n)$ is to consider a general input in the form provided in Eq. (10) and to equate, for the generic term of the nth order, the coefficients of $X_1 X_2 \cdots X_n \exp[(s_1 + s_2 + \cdots + s_n)t]$. As an example, the first aeroelastic Volterra kernel that describes the linear system, obtained by neglecting the nonlinear terms in the aeroelastic governing equations, is obtained by considering the input load as $L_b(t) = X_1 e^{s_1 t}$ [which in dimensionless form is expressed as $l_b(t) = (b/mU_\infty^2)X_1 e^{s_1 t}$]; the response of the system is postulated in the form $h(t) = H_1(s_1)X_1 e^{s_1 t}$ + higher-order terms. Substituting $h(t)$ and its derivatives in the governing equation of motion, one determines the coefficient of $X_1 e^{s_1 t}$.

In a linear aeroelastic formulation the system is completely characterized by a TF $H_1(s_1)$ that contains the aerodynamic term as follows:

$$
\begin{aligned}
H_1(s_1) = \big(&k_{h1} + ms_1^2 + c_{h1}s_1 \\
&+ \underline{s_1\rho C_{L\alpha}bU_\infty C[-is_1 b/U_\infty]} + \tfrac{1}{2}\rho C_{L\alpha}s_1^2 b^2\big)^{-1} \qquad (13)
\end{aligned}
$$

Herein the Theodorsen's function C, connected with the Wagner's indicial function $\phi(\tau)$ via the Laplace's transform as

$$
C(-is) = s\int_0^\infty \phi(\tau)e^{-s\tau}\,\mathrm{d}\tau
$$

has been included in the formulation. The terms underscored by the solid line correspond to the unsteady aerodynamic loads component (circulatory term), whereas the dashed line corresponds to the effect of the added mass. When the aerodynamic loads are neglected and for $s = i\omega$, this result coincides with that of the linear FRF, derived via the conventional modal analysis. The condition $s = i\omega$ corresponds to zero initial conditions; the effect of the initial condition on the nonlinear aeroelastic response can be included via $s = \sigma + i\omega$.

For purely mechanical systems, in the frequency domain the response via Volterra series has been carried out by several authors. In the present study an alternative procedure, based on the multivariable kernel transforms techniques, referred to as higher-order transfer functions (HTFs) is pursued. The two approaches can be correlated to each other, and this is shown also in this work. Assuming zero initial condition, the FRFs are obtained from the TFs by replacing the Laplace transform variable s with $j\omega$.[38]

In the present nonlinear aeroelastic system, toward the estimation of HTFs that are defined as multidimensional Laplace transform of Volterra kernels, a sequence of TFs are employed.

By the use of the linear TF $H_1(s_1)$, the behavior of the linear system is easily determined. It will be necessary to find a complete set of Volterra kernel transforms $H_n(s_1, s_2, \ldots, s_n)$ for nonlinear systems, and for this, in practice, we will use a convergent truncated series. Probing the system with a single harmonic yields only the information about the value of the TFs terms on the diagonal line of the plane s_1, s_2 in the Laplace transformed space, where $s_1 = s_2$. However, to obtain information elsewhere in this space one should use multifrequency excitations. In the same way the second-order Volterra kernel can be determined applying a load depending on two different frequencies expressed as $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t}$. In this case we can express the plunging response in the form

$$h(t) = H_1(s_1)X_1 e^{s_1 t} + H_1(s_2)X_2 e^{s_2 t} + H_2(s_1, s_1)X_1^2 e^{2 s_1 t} + H_2(s_2, s_2)X_2^2 e^{2 s_2 t} + H_2(s_1, s_2)X_1 X_2 e^{(s_1 + s_2)t} + H_2(s_2, s_1)X_2 X_1 e^{(s_2 + s_1)t} + h.o.t. \tag{14}$$

Substituting Eq. (14) in Eq. (8) and equating the terms containing $X_1 X_2 e^{(s_1+s_2)t}$, the second-order aeroelastic Volterra kernel in the Laplace transformed space is obtained as

$$H_2(s_1, s_2) = -(s_1 s_2 c_{h2} + k_{h2}) H_1(s_1) H_1(s_2) H_1(s_1 + s_2) \tag{15}$$

where

$$H_1(s_1 + s_2) = \big\{k_{h1} + (s_1 + s_2)^2 m + c_{h1}(s_1 + s_2) + (s_1 + s_2) \times \rho C_{L\alpha} b U_\infty C[-i(s_1 + s_2)b/U_\infty] + \tfrac{1}{2}\rho C_{L\alpha}(s_1 + s_2)^2 b^2\big\}^{-1} \tag{16}$$

is the first-order Volterra kernel in the Laplace transformed space at the frequency $\omega_1 + \omega_2$ [that is obtained from Eq. (13) in which s_1 is replaced by $s_1 + s_2$]. The terms $H_2(s_1, s_1)$ and $H_2(s_2, s_2)$ can be determined from Eq. (15) replacing s_2 with s_1 and vice versa, respectively. Following the same steps, applying the load $L_b(t) = X_1 e^{s_1 t} + X_2 e^{s_2 t} + X_3 e^{s_3 t}$, equating the terms in the form $X_1 X_2 X_3 \exp[(s_1 + s_2 + s_3)t]$, and remembering that

$$H_1(s_1 + s_2 + s_3) = \big\{k_{h1} + (s_1 + s_2 + s_3)^2 m + c_{h1}(s_1 + s_2 + s_3) + (s_1 + s_2 + s_3)\rho C_{L\alpha} b U_\infty C[-i(s_1 + s_2 + s_3)b/U_\infty] + \tfrac{1}{2}\rho b^2 C_{L\alpha}(s_1 + s_2 + s_3)^2\big\}^{-1} \tag{17}$$

the expressions for the third-order Volterra kernel in the Laplace transformed space results

$$H_3(s_1, s_2, s_3) = -\tfrac{2}{3}(H_1(s_3)\{3H_1(s_1)H_1(s_2)(k_{h3} + c_{h3}s_1 s_2 s_3) + 2H_2(s_1, s_2)[k_{h2} + c_{h2}(s_1 + s_2)s_3]\} + 2\{H_1(s_2)H_2(s_3, s_1)[k_{h2} + c_{h2}s_2(s_1 + s_3)] - H_1(s_2)H_2(s_2, s_3)[k_{h2} + c_{h2}s_1(s_2 + s_3)]\})/[H_1(s_1 + s_2 + s_3)] \tag{18}$$

The constants k_{h2} and c_{h2} multiply the whole expression of H_2, and, if the quadratic nonlinear term in the aeroelastic governing equations is absent, this term vanishes. Herein one of the general properties of Volterra's series is recalled: if all nonlinear terms in the equation of motion for the system consist of odd powers of x and y, then the associated Volterra series have no even-order kernels, and as a result it will possess no even-order TFs. It is also a general property of systems that all TFs can be expressed in terms of H_1. The expressions are functions of the system and can be obtained by using the harmonic probing algorithm. It clearly appears that the HTFs, defined from the Volterra series, are independent of the input to the system.

C. Plunging-Pitching Airfoil Motion in an Incompressible Flowfield

The aeroelastic governing system of an airfoil featuring plunging–twisting coupled motion exposed to a harmonic time-dependent external excitation is

$$m\ddot{h} + S_\alpha \ddot{\alpha} + \sum_{i=1}^{n}\left(c_{hj}\dot{h}^i + k_{hj}h^i\right) - L_a = L_b \tag{19}$$

$$S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + \sum_{i=1}^{n}\left(c_{\alpha j}\dot{\alpha}^i + k_{\alpha j}\alpha^i\right) - M_a = 0 \tag{20}$$

Considering the blast load $L_b(\tau)$ as uniformly distributed in the chordwise direction, no moment contribution $M_b(\tau)$ is introduced in Eq. (20).

Following the steps adopted for one DOF, applying a load depending on one frequency $L_b = X_1 e^{s_1 t}$, and expressing the plunging and pitching displacements in terms of TFs as

$$h(t) = X_1 H_1^h(s_1)e^{s_1 t} + X_1^2 H_2^h(s_1, s_1)e^{2s_1 t} + X_1^3 H_3^h(s_1, s_1, s_1)e^{3s_1 t} \tag{21}$$

$$\alpha(t) = X_1 H_1^\alpha(s_1)e^{s_1 t} + X_1^2 H_2^\alpha(s_1, s_1)e^{2s_1 t} + X_1^3 H_3^\alpha(s_1, s_1, s_1)e^{3s_1 t} \tag{22}$$

the relative kernels and the aeroelastic responses can be determined.

The aeroelastic governing system including the external excitation (such as blast pressure signatures) can be expressed in the Laplace transformed space as

$$s^2\hat{\xi} + \chi_\alpha s^2\hat{\alpha} + 2\zeta_h \frac{\bar{\omega}}{V}s\hat{\xi} + \left(\frac{\bar{\omega}}{V}\right)^2\hat{\xi} + \Im_{nl}^{\xi} + \frac{2}{\mu}\left[s\hat{\alpha} + s^2\hat{\xi} + s^2\left(\frac{1}{2} - a\right)\hat{\alpha}\right]\Phi(s) + \frac{1}{\mu}s^2(\hat{\xi} - a\hat{\alpha}) + \frac{1}{\mu}s\hat{\alpha} = l_b(s) \tag{23}$$

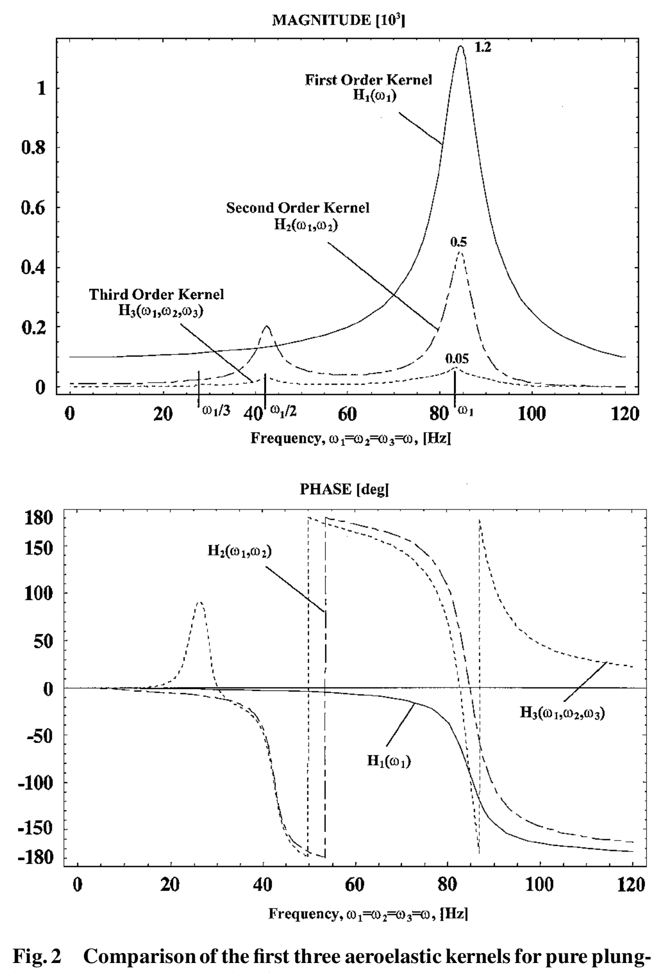

Fig. 2 Comparison of the first three aeroelastic kernels for pure plunging motion. Representation for $s_1 = s_2 = s_3$, that is, $\omega_1 = \omega_2 = \omega_3$.

$$
\begin{aligned}
&\left(\frac{\chi_\alpha}{r_\alpha^2}\right)s^2\hat{\xi}+s^2\hat{\alpha}+\left(\frac{2\zeta_\alpha}{V}\right)s\hat{\alpha}+\frac{\hat{\alpha}}{V^2}+\Im_{nl}^{\alpha}\\
&\quad-\left(\frac{1}{2}+a\right)\frac{2}{\mu}\frac{1}{r_\alpha^2}\left[s\hat{\alpha}+s^2\hat{\xi}+s^2\left(\frac{1}{2}-a\right)\hat{\alpha}\right]\Phi(s)\\
&\quad-\frac{1}{r_\alpha^2}\frac{1}{\mu}as^2(\hat{\xi}-a\hat{\alpha})+\left(\frac{1}{2}-a\right)\frac{1}{r_\alpha^2}\frac{1}{\mu}s\hat{\alpha}+\frac{1}{8}\frac{1}{r_\alpha^2}\frac{1}{\mu}s^2\hat{\alpha}=0
\end{aligned}
\tag{24}
$$

Herein $\hat{(\cdot)}=\mathcal{L}(\cdot)$, and consequently $\hat{\xi}(s)=\mathcal{L}[\xi(t)]$ and $\hat{\alpha}(s)=\mathcal{L}[\alpha(t)]$, whereas $\Im_{nl}^{\xi}$ and $\Im_{nl}^{\alpha}$ are nonlinear functions of ξ and α (and of their derivatives), respectively. Following the same steps, applying the loads $L_b(t)=X_1e^{s_1t}+X_2e^{s_2t}$ and $L_b(t)=X_1e^{s_1t}+X_2e^{s_2t}+X_3e^{s_3t}$, equating the terms in the forms $X_1X_2e^{(s_1+s_2)t}$ and $X_1X_2X_3e^{(s_1+s_2+s_3)t}$, the expressions for the second- and third-order Volterra kernel in the Laplace-transformed space can be obtained.

D. Generalization to Multiple-DOF Systems

The method shown for one- and two-DOF wing sections can be extended to systems featuring multiple-DOF systems in general and to a three-dimensional aircraft wing in particular. The method of deriving the nth order nonlinear aeroelastic TFs is based on the fact that when the aeroelastic system described by the response $y(t)$ (expressed via Volterra series) is excited by a set of k unit amplitude exponentials at the arbitrary frequencies $s_1, s_2, \ldots, s_k$, the output will contain exponential components of the form

$$
\begin{aligned}
y(t)=\sum_{n=1}^{\infty}\sum_{m}\frac{n!}{m_1!m_2!\ldots m_k!}\\
\times H_n(s_1,s_2,\ldots s_k)\exp[(s_1+s_2+\cdots+s_k)t]
\end{aligned}
\tag{25}
$$

where because s_i occurs in $(s_1, s_2, \ldots, s_k)$ m_i times, there are $n!/(m_1!m_2!\ldots m_k!)$ identical terms; m under the summation sign indicates that the sum includes all of the distinct vectors $(m_1, m_2, \ldots, m_k)$ such that

$$\sum_{i=1} m_i = n$$

The presence of nonlinearities causes harmonic excitations and sums of harmonics to appear in the response of the aeroelastic system. Because of the nonlinear formulation, different frequencies can be expected as well.

From an energy point of view, we can observe that $H_1(s_1)$ produces a single-frequency output in response to the simple input e^{s_1t}. However, because the system is nonlinear $H_2(s_1, s_2)$ takes into account the terms that produce an output energy corresponding to the sum of frequencies $\omega_1+\omega_2$, or in other words to the input $e^{(s_1+s_2)t}$. Similarly, the third-order nonlinear aeroelastic kernel will inject a mix of three input frequencies into the total system output (Figs. 2–4). This is the great advantage of this methodology over the other approaches based on the first-order FRF. In contrast to these methodologies, Volterra's series approach is able to capture the transfer of energy between frequencies, which is typical for nonlinear systems.

V. Results and Discussion

For numerical simulations, unless otherwise stated, the following parameters were used: ($m=1$ kg; $b=1$ m; $C_{L\alpha}=2\pi$; $c_{h1}=10$ N/ms^{-1}; $c_{h2}=10$ N/m^{2}s^{-2}; $c_{h3}=10$ N/m^{3}s^{-3}; $k_{h1}=10^4$ N/m; $k_{h2}=10^7$ N/m^2; $k_{h3}=10^8$ N/m^3; $\rho=1.225$ kg/m^3; $U_\infty=1$ m/s). For the two-dimensional wing section experiencing pure plunging only, the first three aeroelastic kernels in magnitude and phase are depicted in Fig. 2 as a function of the frequency, considering the representation along the diagonal of the plane ω_1, ω_2, that is, for $\omega=\omega_1=\omega_2=\omega_3$. As is clearly seen, a reduced influence on the response of the third kernel is experienced.

In Fig. 3 there are depicted the Volterra kernels for the wing section featuring plunging and pitching DOFs. Also in this case the

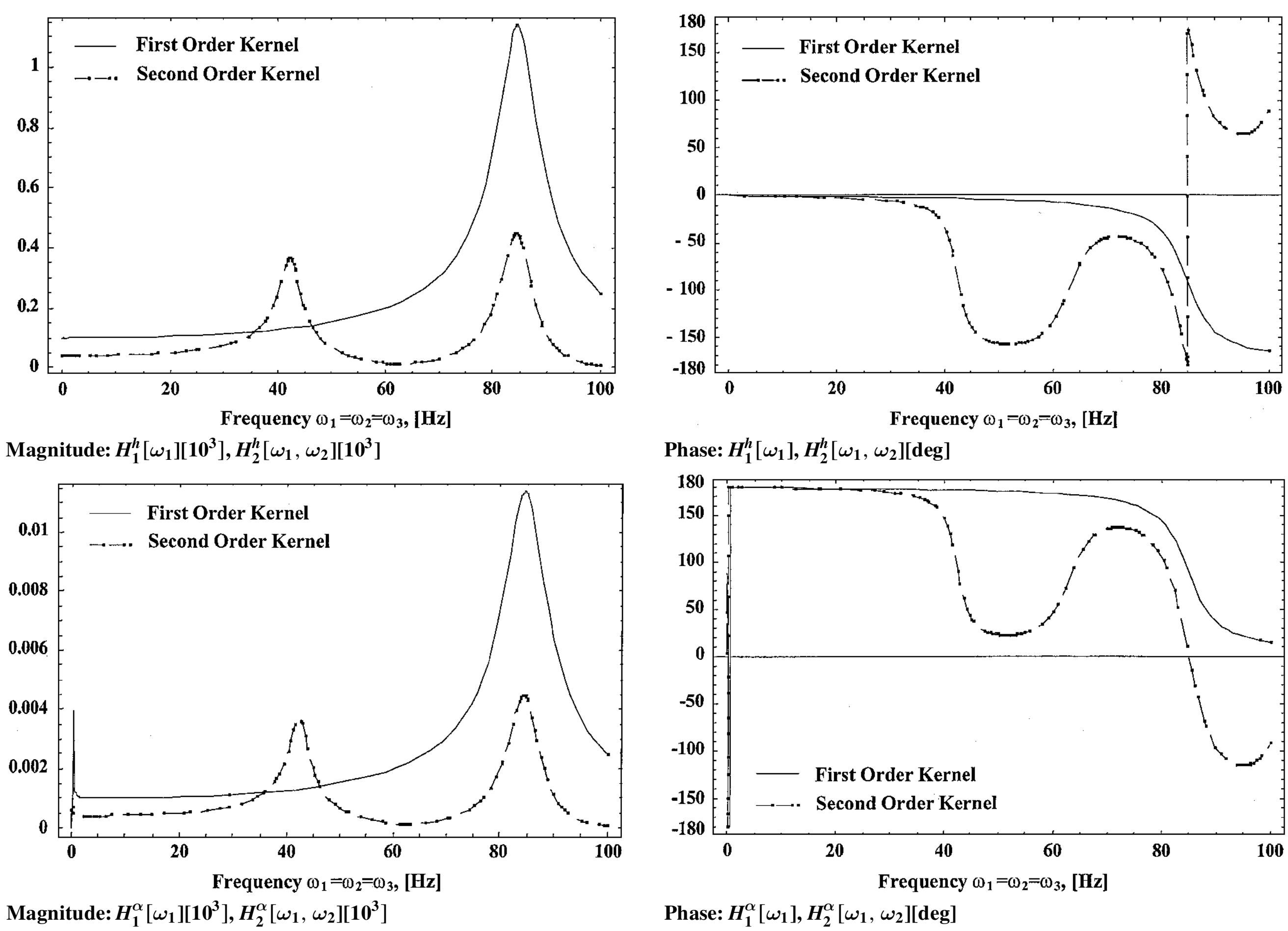

Fig. 3 First two aeroelastic kernels for plunging–pitching coupled motion ($c_{\alpha1}=10$ N/ms^{-1}; $c_{\alpha2}=10$ N/m^{2}s^{-2}; $k_{\alpha1}=10^4$ N/m; $k_{\alpha2}=10^7$ N/m^2; $a=-0.2$m; $U_\infty=0.4U_F$).

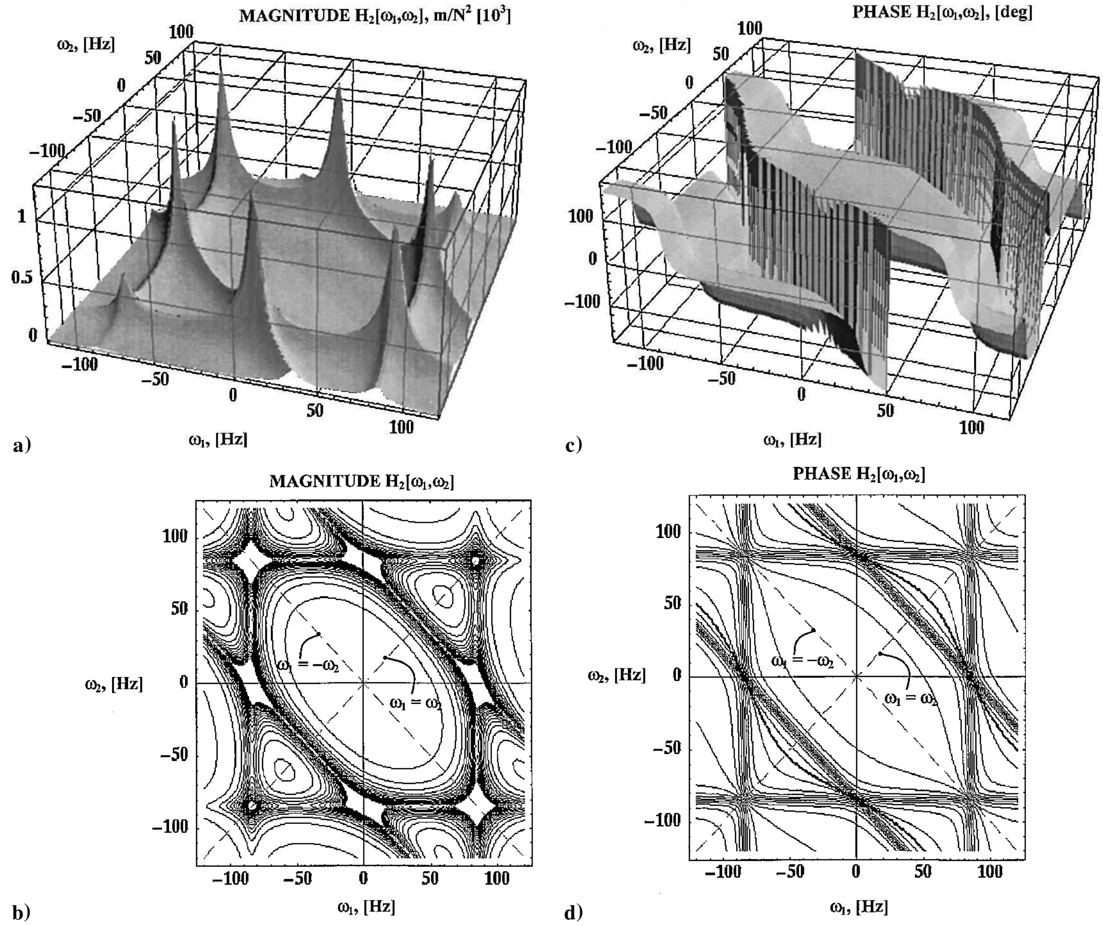

Fig. 4 Three-dimensional and contour plots of second-order aeroelastic kernel.

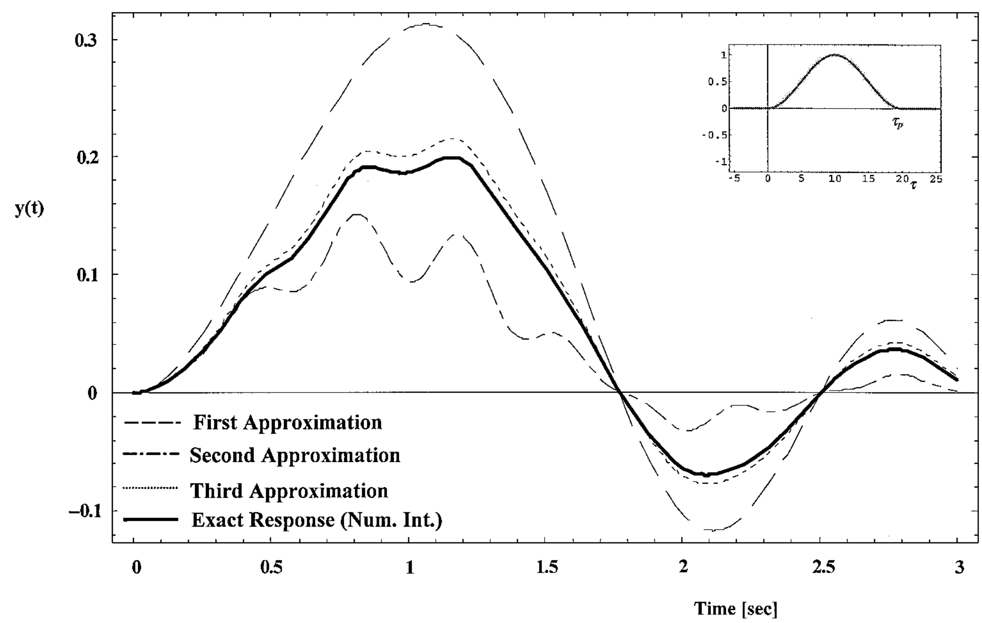

Fig. 5 Convergence study involving the first three kernels and comparison with the exact nonlinear aeroelastic response to a 1-cosine gust pulse, as shown in the inset. Num. Int., Numerical integration.

plots include the magnitude and phase for the kernels in plunging H_i^h and pitching H_i^α, in which i identifies the order of the kernel. In Fig. 4 three-dimensional plots of the magnitude and phase (Figs. 4a and 4c, respectively) of the second-order kernel vs the two frequencies ω_1 and ω_2 are provided. The contour plots (Figs. 4b and 4d, respectively) reveal the symmetry of this kernel with respect to the leading diagonal represented by $\omega_1 = \omega_2$. To assess the versatility and provide a validation of this methodology, a comparison of the predictions of the aeroelastic response of a nonlinear one-DOF wing section using three approximations is shown in Figs. 5 and 6. The excellent agreement of predictions demonstrates both the accuracy of the aeroelastic model and also the power of the methodology based on the Volterra series and indicial function approach. The first-, second-, and third-order approximations of the aeroelastic response to a 1-cosine gust load and triangular blast load are plotted for different parameters, along with the exact response obtained via numerical integration. Both figures reveal the rapid convergence of the approximation. The same approach has been

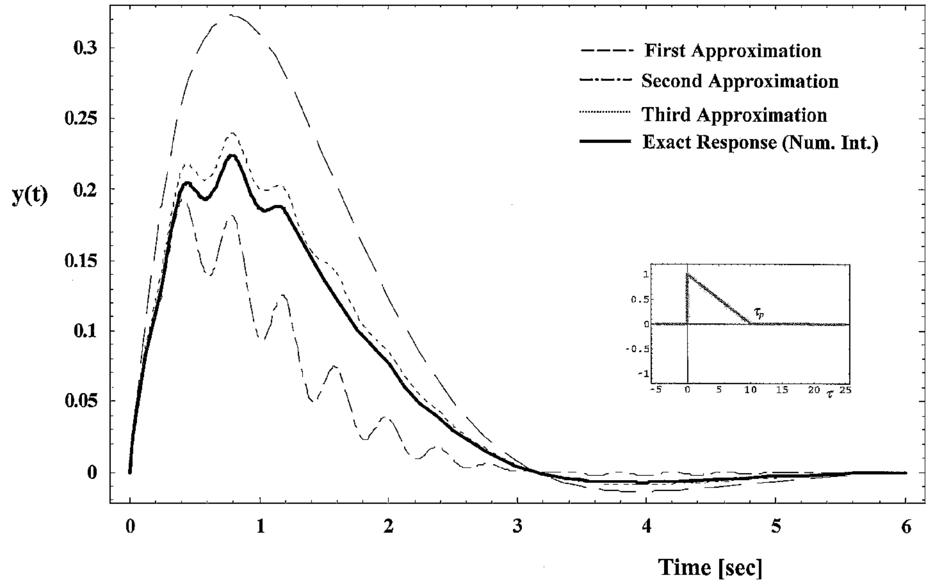

Fig. 6 Convergence study involving the first three kernels and comparison with the exact nonlinear aeroelastic response to a triangular blast load, as shown in the inset. Num. Int., Numerical integration.

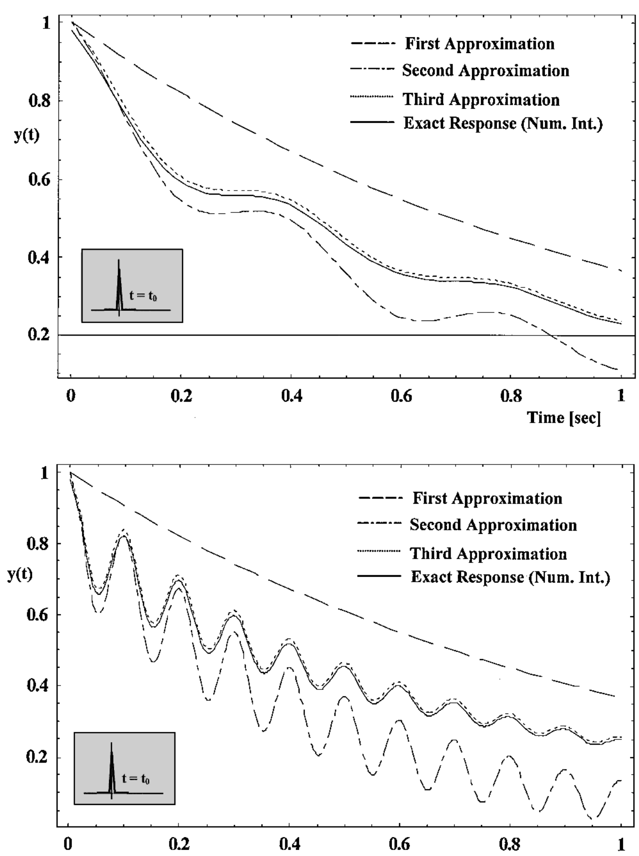

Fig. 7 Convergence study involving the first three kernels and comparison with the exact nonlinear aeroelastic response of the impulse response described in Eq. (26) ($k = 1$; $\omega_0 = 2\pi$). Num. Int., Numerical integration.

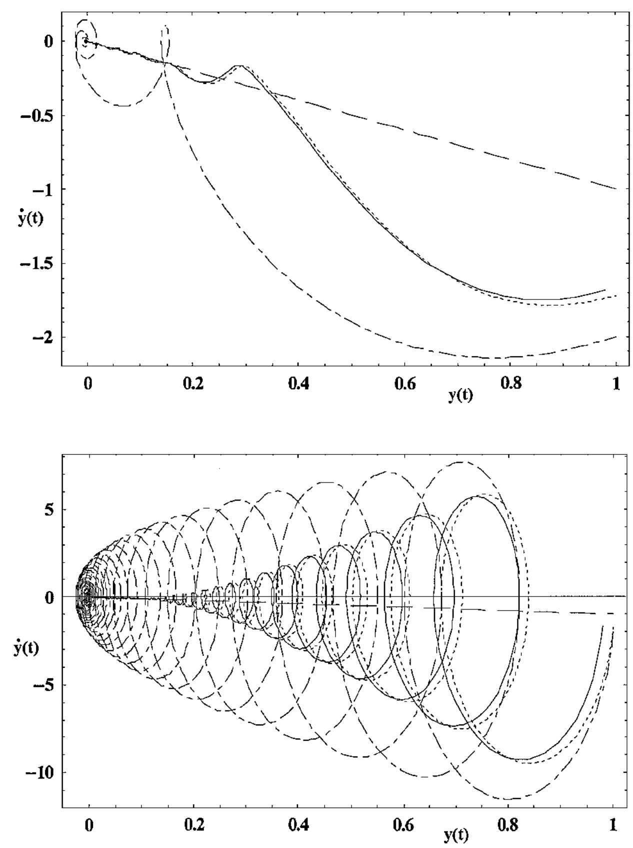

Fig. 8 Convergence study involving the first three kernels and comparison with the exact nonlinear aeroelastic response of the impulse response described in Eq. (26) (phase-space representation) ($k = 10$; $\omega_0 = 20\pi$).

applied to a nonlinear time-varying system represented in Ref. 39 in which a transient response analysis of a continuous system has been addressed via functional techniques and multidimensional Laplace transformation. The impulse response of the system, represented by the differential equation

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} + Ac(t) + km(t)c(t) + \varepsilon c^2(t) = R\delta(t) \tag{26}$$

evaluated with the present analysis, coincides with that shown in the Ref. 39 in which the parameters in use are $A = 1$; $R = 1$; $k = 1, 10$; $\varepsilon = 1$; $m(t) = \sin(\omega_0 t)$; $\omega_0 = 2\pi, 20\pi$. Figures 7 and 8 show the excellent agreement of these two approaches. In Figs. 8, the phase space of the two responses is compared. The results provided in these figures constitute a strong test of the speed of convergence of the present method.[39,40] The coefficients of the nonlinear equation are not small, nor is the period of the time-varying parameter long compared with the natural time constant of the system.

The third-order Volterra kernel in terms of its magnitude and phase for the case in which $\omega_3 = \omega_1$ are depicted in a three-dimensional plot in Figs. 9a and 9c, and the corresponding contour

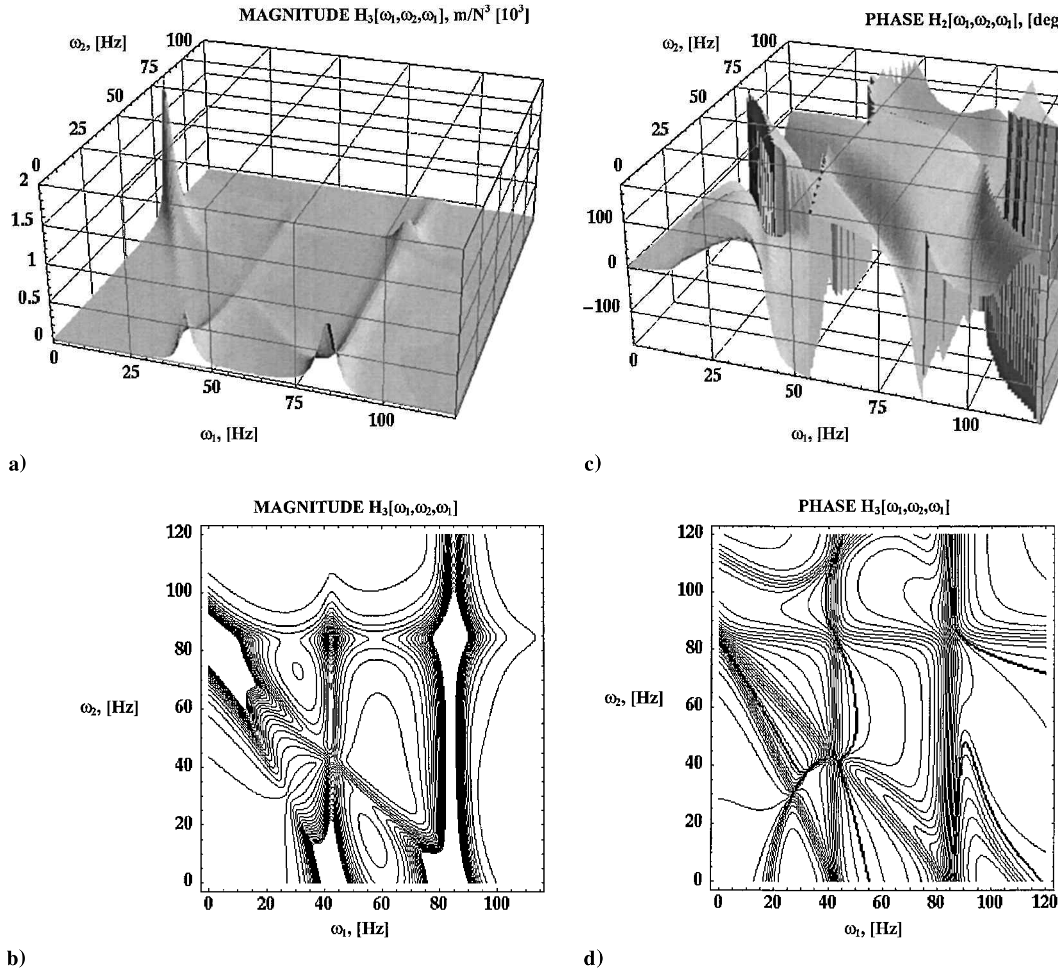

Fig. 9 Three-dimensional and contour plots of third-order aeroelastic kernel.

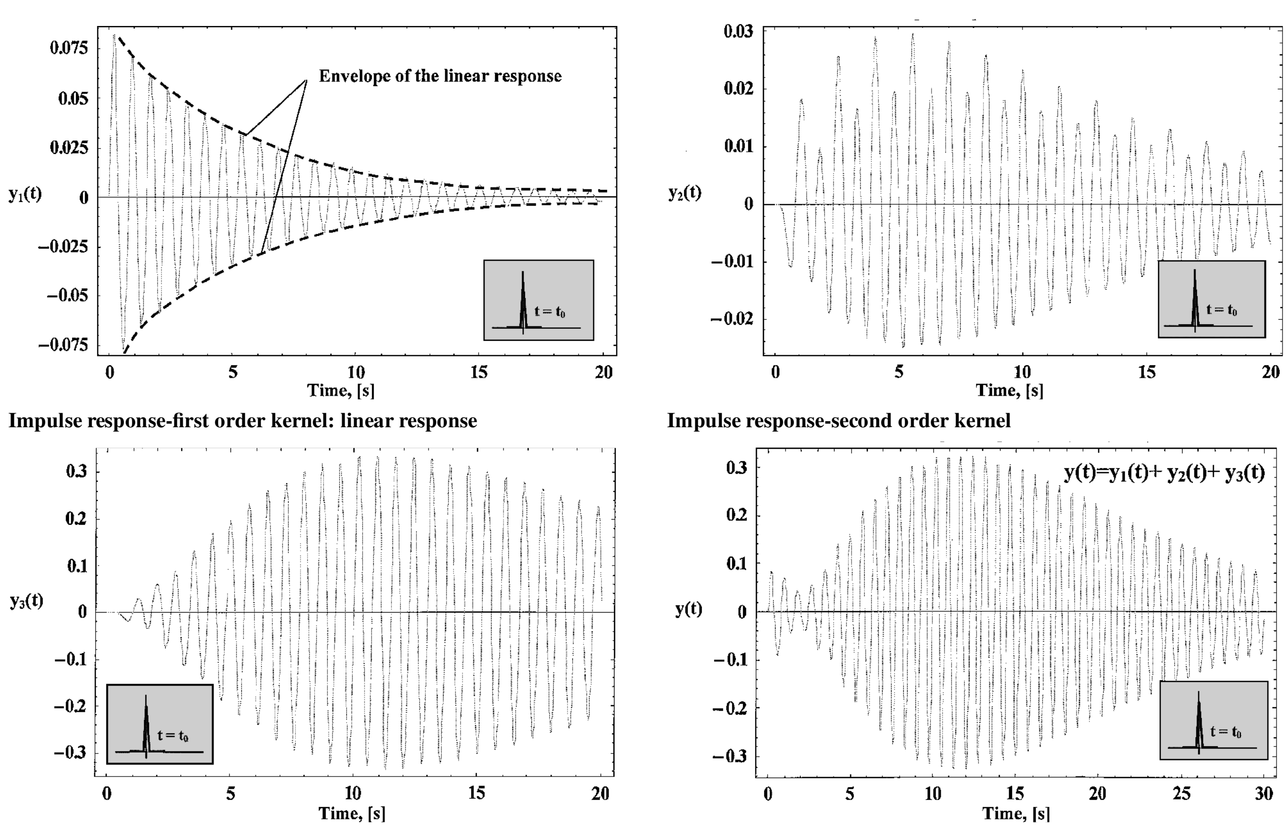

Fig. 10 Time history of the nonlinear aeroelastic response.

plots are presented in Figs. 9b and 9d. The second- and third-order Volterra kernels, in particular, and the nth order kernel, in general, are the second-, third-, and nth-order TFs of the aeroelastic system, where the point $H_n(s_1, \ldots, s_n)$ gives the magnitude and phase of the selected power n output component at frequency of $\omega_1 + \cdots + \omega_n$ as a result of input sinusoids at frequencies $\omega_1, \ldots, \omega_n$. In this sense the high-order Volterra kernels have been determined via multiple sine input. For all of these reasons, these kernels will provide a direct physical interpretation of the high-order terms of the aeroelastic system. From Fig. 4 it appears that the two peaks along the leading diagonal of H_2 correspond to the resonance of the first kernels ω_1 (see Fig. 2) and a resonance $\omega_1/2$, meaning that the effect of the quadratic nonlinearities will be a maximum for excitation frequency of ω_1 and $\omega_1/2$. Consequently in Fig. 9, along the leading diagonal there are three peaks (ω_1, $\omega_1/2$, and $\omega_1/3$), implying that the effect of the cubic nonlinearities will be maximum when the frequency of the sinusoidal input coincides with these frequencies. In addition, as it appears clearly from Fig. 2, the magnitude of the second and third kernels decreases rapidly (peak of H_2 is 2.4 times smaller then H_1 and the peaks of H_3 is 10 times smaller than H_2), implying that accurate results can be obtained with a few kernels only.

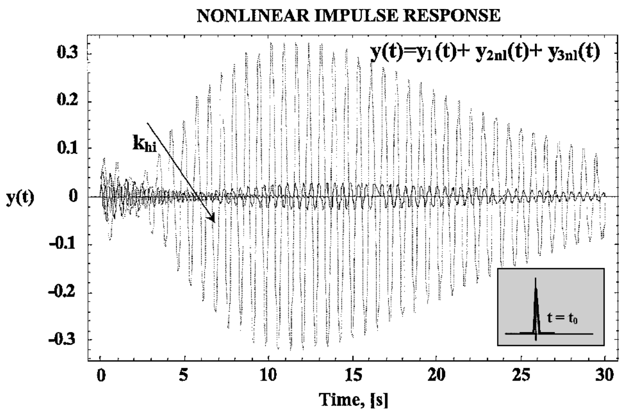

Fig. 11 Influence of the linear stiffness coefficient k_{h1} on the nonlinear aeroelastic response; parameters as in Fig. 3.

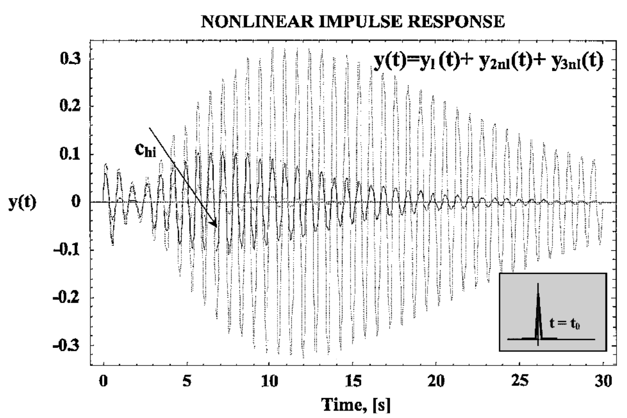

Fig. 12 Influence of the linear damping coefficient c_{h1} on the nonlinear aeroelastic response; parameters as in Fig. 3.

Determination of the subcritical aeroelastic response to any time-dependent externally applied load is useful in the design of wing structures and of the associated feedback control systems. This response can be determined by use of the complex inversion formula from the frequency domain to the time domain. Another idea is to find the one-dimensional original response after the identification of all of the variables from the n-dimensional Laplace transformed space.[14,41] This case can be represented as

$$g(\tau) \equiv h_n(\tau_1, \tau_2, \ldots \tau_n)|_{\tau_1 = \tau_2 = \cdots = \tau_n = \tau} \tag{27}$$

Because $h_n = \mathcal{L}^{-1}[H_n]$, $g(\tau)$ is obtained using $G(s) = \mathcal{L}[g(\tau)]$; $G(s)$ can be calculated by means of an $(n-1)$-dimensional inversion formula. The function $g(\tau)$ has a corresponding Laplace transform $G(s)$ (also called *associated transform* of H_n) in the single-dimensional Laplace-transformed space.

The response in time can be obtained from $H(s_1, s_2, \ldots, s_n)$ by determining $G(s)$ first and evaluating the one-dimensional inverse Laplace transform $g(\tau)$. This approach is called *association of variable*.[14,41] Using this concept, the nonlinear aeroelastic response in the time domain is depicted in Figs. 10 for a one-DOF wing section featuring the plunging DOF. In this figure the first plot represents the linear impulse response that corresponds to the convolution integral for the linear analysis. The other three plots represent the components of the response caused by the second- and the third-order kernels and the total response as a combination of the three partial responses. The influence of the linear stiffness and of the damping coefficients on the response are displayed in Figs. 11 and 12. An increase of those coefficients contributes to the decrease of the magnitude of the kernels and, consequently, of the response amplitude. Results not shown in this paper reveal that the geometrical nonlinearities contribute to a limited growth of the deflection in the postflutter range and so to the avoidance of the occurrence of

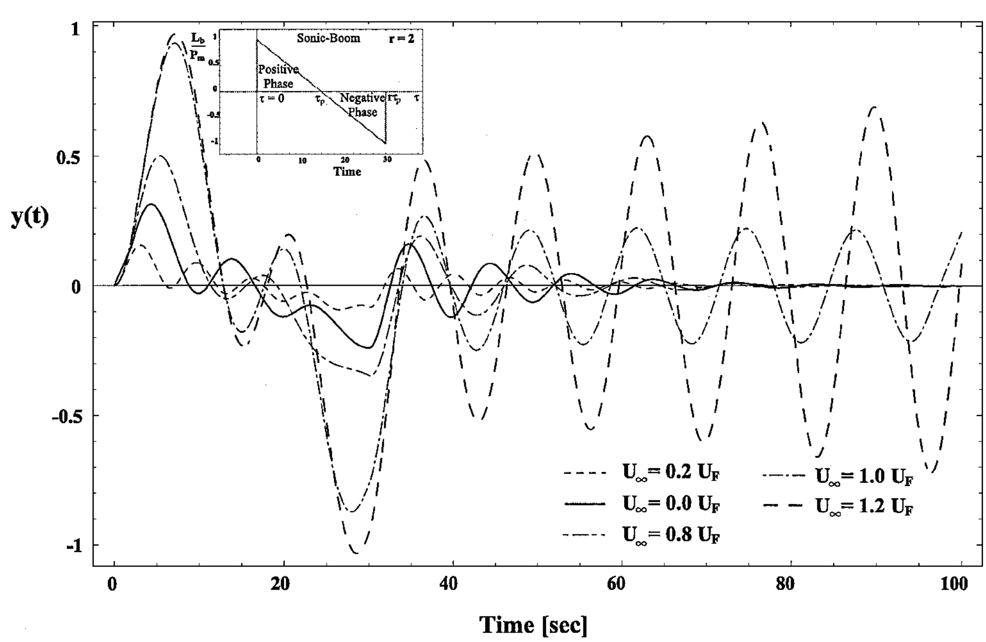

Fig. 13 Influence of the flight speed on the nonlinear aeroelastic response to a sonic boom ($\tau_p = 15s$; $r = 2$), as shown in the inset, evaluated with three kernels; parameters as in Fig. 3.

catastrophic failures.[16] Figure 13 highlights the effect of the speed parameter U_∞ on the wing sections subjected to sonic-boom pressure signature as shown in the inset. Herein τ_p denotes the positive phase duration of the pulse measured from the time of impact of the structure. For $r = 2$ the N-shaped pulse degenerates into a symmetric sonic-boom pulse, in the sense that its positive phase has the same characteristics as its negative one, and for $r = 1$ a triangular pulse that corresponds to an explosive pulse is obtained. It becomes apparent that the amplitude of the response time history (that have been evaluated for practical use with three kernels) increases with the increase of U_∞. As it clearly appears, the plunging amplitudes at zero flight speed are slightly larger than those emerging at a small flight speed ($U_\infty = 0.2U_F$). This is caused by the fact that, in contrast to the former case when the aerodynamic damping term is zero, in the latter case the aerodynamic damping that is associated with this flight case contributes toward attenuating the oscillations. However, this trend is reversed when the flight speed further increases, and in such a case larger amplitudes are experienced toward the flutter speed. Moreover, for the speed parameter U_∞ greater than that corresponding to the flutter conditions (these ones determined within the linearized aeroelastic system), as expected, the response becomes unbounded.

VI. Conclusions

In this paper several issues related to the approach of the nonlinear aeroelastic response via Volterra series approach have been presented. Following the same approach presented here, the character of the instability boundary, that is, benign or catastrophic, can also be addressed. It was also shown that the method based on Volterra series provides a unified and efficient approach for addressing nonlinear aeroelastic phenomena. Moreover, this approach can be extended as to include also active control capabilities. Comparison of results carried out via Volterra series in conjunction with indicial functions approach and classical approach has been provided for the linearized model. For the fully nonlinear model, a comparison with the exact numerical simulation of the nonlinear aeroelastic response has been provided. The aerodynamic indicial functions (for incompressible/compressible flowfields) considered in conjunction with Volterra's series approach can be used as a powerful analytical tool for developing unsteady aerodynamic models and a unified nonlinear aeroelastic model.

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